SOME APPLICATIONS OF DIRECT INTEGRAL DECOMPOSITIONS OF W^*-ALGEBRAS[1]

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ABSTRACT. Let $\mathcal{C}$ be a W^*-algebra and let $A \in \mathcal{C}$. $\mathcal{K}(\mathcal{C})$ and $C(A)$ represent certain convex subsets of $\mathcal{C}$. We prove the following via direct integral theory:

(1) If $\mathcal{C}$ is of type I_∞, II_∞, or III, then $C(A) = \{0\}$ iff $A \in \mathcal{K}(\mathcal{C})$.

(2) If $\mathcal{C}$ is of type I or II, then $\mathcal{K}(\mathcal{C})$ is strongly dense in $\mathcal{C}$.

(3) If $\mathcal{C}$ is of type I_∞, II_∞, or III and $\mathcal{B}$ is a W^*-subalgebra of $\mathcal{C}$, we give sufficient conditions for a Schwartz map P of $\mathcal{C}$ into $\mathcal{B}$ to annihilate $\mathcal{K}(\mathcal{C})$.

Several preliminary lemmas that are useful for direct integral theory are also proved.

1. Introduction. Let $\mathcal{C}$ be a W^*-algebra acting in a Hilbert space h (always assumed separable). Recall that $\mathcal{C}'$ denotes the commutant of $\mathcal{C}$ and is also a W^*-algebra. By the center of $\mathcal{C}$, we mean the abelian W^*-algebra $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$. $\mathcal{C}_1$ denotes the unit ball of $\mathcal{C}$ while the symbols $\mathcal{U}(\mathcal{C})$, $\mathcal{F}(\mathcal{C})$ and $\mathcal{P}(\mathcal{C})$ denote those collections of operators which are, respectively, the unitaries, nonzero finite projections and all projections belonging to $\mathcal{C}$. We use the symbol $\mathcal{S}(\mathcal{C})$ to denote the set of all nonnegative real valued functions defined on $\mathcal{U}(\mathcal{C})$ with finite support such that $\Sigma_{U \in \mathcal{U}(\mathcal{C})} f(U) = 1$.

For each $A \in \mathcal{C}$, $CO(A)$ denotes the convex hull of the set $\{UAU^* | U \in \mathcal{U}(\mathcal{C})\}$ and by $\overline{CO(A)}$, we mean its weakly closed convex hull. By [3, Lemma V.2.4], $\overline{CO(A)}$ is the weak closure of $CO(A)$. We define $C(A)$ by $C(A) = \overline{CO(A)} \cap \mathcal{Z}(\mathcal{C})$. Since every W^*-algebra is a weakly closed convex set it is clear that $C(A)$ is also a weakly closed convex set. By [3, Corollary VI.1.5], $\overline{CO(A)}$ and $C(A)$ are also strongly closed. Also for each $A \in \mathcal{C}$ and each $f \in \mathcal{S}(\mathcal{C})$, we define $f \cdot A$ by $f \cdot A = \Sigma_{U \in \mathcal{U}(\mathcal{C})} f(U)UAU^*$. Clearly, $f \cdot A$ defines a convex combination of elements of the type UAU^*. Thus if $B \in \overline{CO(A)}$, then there exists a countable sequence of functions $\{f_n\}$ contained in $\mathcal{S}(\mathcal{C})$ such that $\{f_n \cdot A\}$ converges strongly to B. We shall use this fact in the proofs that follow.

When h is a direct integral Hilbert space we use the symbol h_∞ to denote the underlying Hilbert space of h and write $h = \int_\Lambda \oplus h_\infty \mu(d\lambda)$. Corollary I.5.10 of [4]

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establishes that every global[2] W^*-algebra could be decomposed as a direct integral of factors relative to some direct integral Hilbert space for which we may assume the complete separable metric space Λ is the reals. It is understood that μ is a finite positive regular Borel measure on Λ. When $\mathcal{Q}$ is decomposed as a direct integral, we write $\mathcal{Q} = \int_\Lambda \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$. Theorem I.6.1 of [4] establishes the uniqueness of this representation and [4, Lemma I.5.2] shows that each operator $A \in \mathcal{Q}$ is decomposable and we write $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$. When the μ-measurable function $A(\lambda)$ has the special form $m(\lambda)I$, where $m(\lambda)$ is a μ-a.e. bounded Borel measurable scalar valued function, we say that A is a *diagonal operator*. For detailed definitions of direct integral Hilbert spaces and direct integral decompositions, we refer the reader to [4, Definitions I.2.5 and I.5.1].

Now let $\mathcal{C}_1(\mathcal{B})$ denote the trace-class in a factor $\mathcal{B}$ (see [4, Theorem II.3.12]) and let $\mathcal{Q} = \int_\Lambda \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of a W^*-algebra into factors. Then the set $\mathcal{K}(\mathcal{Q})$ is defined to be all those operators $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$ $\in \mathcal{Q}$ such that $A(\lambda) \in \mathcal{C}_1(\mathcal{Q}(\lambda))$ μ-a.e. By [4, Lemma I.3.1 and Theorem II.3.12], we observe that $\mathcal{K}(\mathcal{Q})$ is convex. Also if $\mathcal{Q}$ is of type III, then $\mathcal{K}(\mathcal{Q}) = \{0\}$ since $\mathcal{C}_1(\mathcal{Q}(\lambda)) = \{0\}$ μ-a.e.

Both concepts $C(A)$ and $\mathcal{C}_1(\mathcal{B})$ have had important application in the development of W^*-algebra theory. Schwartz used the fundamental construction defining $C(A)$ to introduce property P and effectively applied this invariant to distinguish a third isomorphism class of type II_1 factors (see [4, pp. 168–173]). In [1], Conway presented an analysis of $C(A)$ and $\mathcal{C}_1(\mathcal{B})$ in relationship to infinite factors. He proved that $C(A) = \{0\}$ iff $A \in \mathcal{C}_1(\mathcal{B})$, for a factor $\mathcal{B}$ of infinite type. The basic ideas of this paper were motivated by his work. It is our aim to extend Conway's theorem on factors to global W^*-algebras and then to apply the latter result to a situation involving Schwartz maps. In addition to the theorem on density, we also prove several preliminary lemmas, some of which are of independent interest.

2. Statements and proofs of preliminary lemmas. We begin by recalling some essential results from [4] concerning analytic sets.

DEFINITION [4, I.4.1]. A subset E of a topological space X is *analytic* iff there exists a complete separable metric space M and a continuous function of M onto E.

LEMMA [4, I.4.2]. *The union and intersection of a countable sequence of analytic sets are analytic.*

COROLLARY [4, I.4.3]. *If X is a separable metric space which is the union of a countable sequence of complete subsets, then every Borel subset of X is analytic.*

LEMMA [4, I.4.4]. *If X and Y are topological spaces, E is an analytic subset of $X \times Y$ and π is the projection of $X \times Y$ onto X, then $\pi(E)$ is analytic.*

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[2] Global W^*-algebras are nonfactors. Such an algebra is of a given type iff all of the factors in its direct integral decomposition are of that type (see [4, Definition III.1.4]).

LEMMA [4, I.4.6]. *Let μ be a positive Borel measure on a separable metric space X and suppose X differs by a μ-null set from the union of a countable sequence of increasing compact subsets of finite measure. Then every analytic subset of X is μ-measurable.*

We now state the important *Principle of Measurable Choice* which we shall always refer to as P.M.C.

LEMMA [4, I.4.7]. *Let Y be a complete separable metric space and let μ and X be as in the preceding lemma. Let E be an analytic subset of $X \times Y$ with $F = \pi(E)$. Then there exist Borel subsets F_1 and F_2 of X and a mapping g of F_1 into Y such that:*

(a) $\mu(F_2) = 0$,

(b) F_1 *is contained in F, F is contained in $F_1 \cup F_2$ and $F_1 \cap F_2$ is empty,*

(c) $g^{-1}(U)$ *is a Borel set for each open subset U of Y, i.e., g is μ-measurable,*

(d) $(x, g(x)) \in E$ *for each $x \in F_1$.*

In addition to the weak and the strong operator topologies for $B(h)$, the space of all bounded operators acting in a Hilbert space h, we will need the *strong-* operator topology* which we define by the following subbasis for open sets (see [4, Definition I.4.10]):

$N(t, x, \varepsilon) = \{A \mid |(A - T)x| < \varepsilon$ and $|(A^* - T^*)x| < \varepsilon\}$, where the positive number ε, $T \in B(h)$ and $x \in h$ are chosen arbitrarily. We note that adjunction is continuous in the strong-* operator topology. While addition is bicontinuous on arbitrary sets, multiplication is bicontinuous on bounded sets and continuous on arbitarary sets only if one of the variables is held fixed.

The following theorem of Lusin, which we state here for convenience, will be cited often in the proofs that follow.

LEMMA [3, XII.3.17]. *Let μ be a finite positive regular measure on the Borel sets of a topological space X. Then, for every Banach space valued μ-measurable function f on X and every $\varepsilon > 0$, there is an open Borel set E in X with $\mu(E) < \varepsilon$ and such that the restriction of f to the complement of E is continous.*

To illustrate our use of Lusin's lemma, let the countable sequence of positive numbers $\{\varepsilon_n = 1/n\}$ be given and suppose $\{A_i = \int_\Lambda \oplus A_i(\lambda)\mu(d\lambda)\}$ is a countable sequence of decomposable operators. By [4, Definition I.2.5] each operator valued function $A_i(\lambda)$ is weakly measurable and, since the strong-* operator topology is finer than the weak operator topology, it follows from [3, Theorem III.6.11], that they are strong-* measurable. Let f_i denote the operator valued function $A_i(\lambda)$ and apply Lusin's lemma to find an open subset $E(i)_{\varepsilon_1}$ of Λ such that the restriction of f_i to $\tilde{E}(i)_{\varepsilon_1} = \Lambda - E(i)_{\varepsilon_1}$ is strong-* continuous and $\mu(E(i)_{\varepsilon_1}) < \varepsilon_1 2^{-i}$. Write $E_{\varepsilon_1} = \bigcap_{i=1}^{\infty}\tilde{E}(i)_{\varepsilon_1}$ and put $\hat{E}_{\varepsilon_1} = \Lambda - E_{\varepsilon_1}$. Clearly, E_{ε_1} is a closed subset of Λ to which the restriction of each f_i is strong-* continuous and $\mu(\hat{E}_{\varepsilon_1}) < \varepsilon_1$. Applying Lusin's lemma once again, we find that the subspace $\hat{E}_{\varepsilon_1}$ taken with its relative topology contains a relatively closed subset E_{ε_2} to which the restriction of each f_i is strong-* continuous and a relatively open subset $\hat{E}_{\varepsilon_2} = \hat{E}_{\varepsilon_1} - E_{\varepsilon_2}$ such that $\mu(\hat{E}_{\varepsilon_2}) < \varepsilon_2$. Since $\hat{E}_{\varepsilon_1}$ is open in Λ, it follows that each relatively open subset of $\hat{E}_{\varepsilon_1}$ is also open in Λ. Thus

$\Lambda - \hat{E}_{\epsilon_2}$ is closed in Λ. We have $\Lambda - \hat{E}_{\epsilon_2} = E_{\epsilon_2} \cup E_{\epsilon_1}$, where $E_{\epsilon_2} \cap E_{\epsilon_1} = \varnothing$. Since

$$\left(f_i | \Lambda - \hat{E}_{\epsilon_2} \right)^{-1}(U) = \left(f_i | E_{\epsilon_2} \right)^{-1}(U) \cup \left(f_i | E_{\epsilon_1} \right)^{-1}(U)$$

for each open set of operators U contained in $B(h_\infty)$, it follows readily that the restriction of each f_i to $\Lambda - \hat{E}_{\epsilon_2}$ is strong-* continuous. Continuing in this manner, we see that Λ contains a countable sequence of decreasing open subsets $\{b_n = \hat{E}_{\epsilon_n}\}$ and a countable sequence of increasing closed subsets $\{a_n = \Lambda - b_n\}$ such that $\mu(\Lambda - \bigcup_{n=1}^{\infty} a_n) = \mu(\bigcap_{n=1}^{\infty} b_n) = 0$ and such that each f_i is strong-* continuous on each set a_n.

Now we shall prove several lemmas that will be needed in the sequel.

LEMMA 1. *Let $\mathcal{C} = \int_\Lambda \oplus \mathcal{C}(\lambda)\mu(d\lambda)$ be a W*-algebra acting in h and let S denote $B(h_\infty)_1$ taken with the strong-* operator topology. Let k be a positive integer and let $S^{(k)}$ denote the k-fold Cartesian product of S with itself. Equip $S^{(k)}$ with the product topology arising from the topology on S. Then if N is a Borel subset of Λ, the set $F = \{(\lambda, T) | \lambda \in N, T \in (\mathcal{C}(\lambda) \cap S)^{(k)}\}$ is a Borel subset of $\Lambda \times S^{(k)}$.*

PROOF. By [4, Lemma I.4.11], S is a complete separable metric space and so is $S^{(k)}$. Let d denote the metric which defines the product topology on $S^{(k)}$. By [6, Lemma I.5(c)], $\mathcal{C}$ contains a countable sequence of operators $\{B_n\}$ such that $\{B_n(\lambda)\}$ is strong-* dense in $\mathcal{C}(\lambda)_1$ μ-a.e. By Lusin's lemma, Λ differs by a Borel null set e from the union of a countable sequence of increasing closed subsets $\{e_i\}$ such that each $B_n(\lambda)$ is strong-* continuous on each set e_i. Then for μ-a.a. λ the set $\{B_n(\lambda)\}^{(k)}$ is dense in $(\mathcal{C}(\lambda) \cap S)^{(k)}$ relative to the product topology on $S^{(k)}$ and since this set is countable, we may write it as a countable sequence $\{A_m(\lambda) = (B_{n_1}(\lambda), B_{n_2}(\lambda), \ldots, B_{n_k}(\lambda))\}$ contained in $\{B_n(\lambda)\}^{(k)}$. Since each $B_n(\lambda)$ is strong-* continuous on each set e_i, so are the $A_m(\lambda)$.

Define subsets $F(i, j, m)$ of $\Lambda \times S^{(k)}$ as sets of all pairs (λ, T) satisfying the following conditions:

(a) $\lambda \in N \cap e_i$,

(b) $d(T, A_m(\lambda)) \leq 1/j$.

Condition (a) defines a Borel set. Condition (b) defines a closed set. Thus $F(i, j, m)$ is a Borel subset of $\Lambda \times S^{(k)}$ and so is $F = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} F(i, j, m)$. Q.E.D.

Hereafter, S, e and the e_i are as in Lemma 1.

LEMMA 2. *Let $\mathcal{C} = \int_\Lambda \oplus \mathcal{C}(\lambda)\mu(d\lambda)$ be a W*-algebra acting in h. Then if N is a Borel subset of Λ, the set $F = \{(\lambda, U) | \lambda \in N, U \in \mathcal{U}(\mathcal{C}(\lambda))\}$ is a Borel subset of $\Lambda \times S$.*

PROOF. Define subsets $F(i)$ of $\Lambda \times S$ as sets of all pairs (λ, U) satisfying the following conditions:

(a) $\lambda \in N \cap e_i$,

(b) $U \in \mathcal{C}(\lambda) \cap S$,

(c) $UU^* = U^*U = I$.

Condition (a) defines a Borel set. By Lemma 1, conditions (a) and (b) define a Borel set. Condition (c) defines a closed set and shows that U is unitary. Hence each $F(i)$ is a Borel subset of $\Lambda \times S$ and so is $F = \cup_{i=1}^{\infty} F(i)$.   Q.E.D.

LEMMA 3. *Let $\mathcal{Q} = \int_{\Lambda} \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ be a W^*-algebra acting in h. Then there exists a countable sequence $\{U_n\}$ contained in $\mathfrak{U}(\mathcal{Q})$ such that for μ-a.a. λ the set $\{U_n(\lambda)\}$ is strong-* dense in $\mathfrak{U}(\mathcal{Q}(\lambda))$.*

PROOF. Define subsets $G(i)$ of $\Lambda \times S$ as sets of all pairs (λ, U) satisfying the following conditions:

(a) $\lambda \in e_i$,

(b) $U \in \mathfrak{U}(\mathcal{Q}(\lambda))$.

By Lemma 2, each $G(i)$ is a Borel subset of $\Lambda \times S$. Since S is a separable metric space, there exists a countable sequence of open sets $\{W_n\}$ contained in S which forms a basis for the topology on S. Let $F(i, n) = \{(\lambda, U) \in G(i) | U \in W_n\}$. Clearly, each $F(i, n)$ is a Borel set and so is $F(n) = \cup_{i=1}^{\infty} F(i, n)$. By [4, Lemma I.4.3], $F(n)$ is analytic. Thus by P.M.C., there exists a countable sequence of Borel null sets $M(n)$ and a countable sequence of μ-measurable mappings Y_n of $H(n) = \pi(F(n)) - M(n)$ into S such that $(\lambda, Y_n(\lambda)) \in F(n)$ for each $\lambda \in H(n)$. Put $Y_n(\lambda) = 0$ for $\lambda \notin H(n)$ and define μ-measurable operator valued functions $U_n(\lambda)$ by $U_n(\lambda) = Y_n(\lambda)$. Then by [4, Definition I.2.5], we may write $U_n = \int_{\Lambda} \oplus U_n(\lambda)\mu(d\lambda)$ and by [4, Lemmas I.3.1 and I.5.2], we know that $\{U_n\}$ is contained in $\mathfrak{U}(\mathcal{Q})$.

Now we shall show that $\{U_n(\lambda)\}$ is strong-* dense in $\mathfrak{U}(\mathcal{Q}(\lambda))$ μ-a.e. Since $\mathfrak{U}(\mathcal{Q}(\lambda))$ is contained in S, the set $\{K_n(\lambda) = \mathfrak{U}(\mathcal{Q}(\lambda)) \cap W_n\}$ is a basis for $\mathfrak{U}(\mathcal{Q}(\lambda))$. Let $D = e \cup \cup_{n=1}^{\infty} M(n)$ and suppose $\lambda \notin D$. Then if $K_n(\lambda)$ is not empty, $\lambda \in H(n) - D$ and $U_n(\lambda) \in K_n(\lambda)$ also. It follows that for each $\lambda \in H(n) - D$, $U_n(\lambda) \in K_n(\lambda)$ so that the sequence $\{U_n(\lambda)\}$ is strong-* dense in $\mathfrak{U}(\mathcal{Q}(\lambda))$ μ-a.e. Q.E.D.

LEMMA 4. *Let $\mathcal{Q} = \int_{\Lambda} \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ be a W^*-algebra acting in h, and let $A \in \mathcal{Q}$. Then $B \in C(A)$ iff B is decomposable and $B(\lambda) \in C(A(\lambda))$ μ-a.e.*

PROOF. We shall first show that $B \in \mathfrak{Z}(\mathcal{Q})$ iff B is decomposable and $B(\lambda) \in \mathfrak{Z}(\mathcal{Q}(\lambda))$ μ-a.e. Since $\mathcal{Q}$ is decomposable, $\mathcal{Q}'$ is decomposable and we may write $\mathcal{Q}' = \int_{\Lambda} \oplus \mathcal{Q}'(\lambda)\mu(d\lambda)$ by [4, Lemma I.5.7]. Then by [4, Lemma I.5.8], $\mathcal{Q} \cap \mathcal{Q}'$ is also decomposable and $\mathcal{Q} \cap \mathcal{Q}' = \int_{\Lambda} \oplus \mathcal{Q}(\lambda) \cap \mathcal{Q}'(\lambda)\mu(d\lambda)$. By [4, Corollary I.5.3], $\mathfrak{Z}(\mathcal{Q}) = \mathcal{Q} \cap \mathcal{Q}'$ iff $\mathfrak{Z}(\mathcal{Q})(\lambda) = \mathfrak{Z}(\mathcal{Q}(\lambda))$ μ-a.e. Thus $B \in \mathfrak{Z}(\mathcal{Q})$ iff B is decomposable and $B(\lambda) \in \mathfrak{Z}(\mathcal{Q}(\lambda))$ μ-a.e. by [4, Lemma I.5.2].

Now let $B \in C(A)$. Then $B \in \mathfrak{Z}(\mathcal{Q})$. Thus B is decomposable and $B(\lambda) \in \mathfrak{Z}(\mathcal{Q}(\lambda))$ μ-a.e. by the preceding paragraph. Also $B \in \overline{CO}(A)$ and there exists a countable sequence of functions $\{f_n\}$ contained in $\mathfrak{S}(\mathcal{Q})$ such that $\{f_n \cdot A\}$ converges strongly to B. Then by [4, Lemma I.3.1], $f_n \cdot A$ is decomposable for each n and $(f_n \cdot A)(\lambda) = \Sigma_{U \in \mathfrak{U}(\mathcal{Q})} f_n(U)U(\lambda)A(\lambda)U(\lambda)^*$. Thus for each n, $(f_n \cdot A)(\lambda) \in CO(A(\lambda))$ μ-a.e. By [5, p. 443], $\{f_n \cdot A\}$ contains a subsequence $\{f_{n_i} \cdot A\}$ such that $\{(f_{n_i} \cdot A)(\lambda)\}$ converges strongly to $B(\lambda)$ μ-a.e. Thus $B(\lambda) \in \overline{CO}(A(\lambda))$ μ-a.e. and it follows that $B(\lambda) \in C(A(\lambda))$ μ-a.e.

Conversely, suppose B is decomposable and $B(\lambda) \in C(A(\lambda))$ μ-a.e. Then $B(\lambda) \in \mathcal{Z}(\mathcal{C}(\lambda))$ μ-a.e. and $B \in \mathcal{Z}(\mathcal{C})$ by the first paragraph of the present proof. It remains to show that there exists a countable sequence of functions $\{t_m\}$ contained in $\mathcal{E}(\mathcal{C})$ such that $\{t_m \cdot A\}$ converges strongly to B. Then $B \in \overline{CO}(A)$ and it follows that $B \in C(A)$.

To prove the assertion, we argue as in [7]. Let d denote the metric which defines the topology on S and define the function W by $W(A) = d(A, 0)$. Then for a bounded sequence $\{A_n\}$, we have that $\{A_n\}$ converges strong-* to zero iff $W(A_n)$ converges to zero. By Lemma 3, there exists a countable sequence $\{U_n\}$ contained in $\mathcal{U}(\mathcal{C})$ such that for μ-a.a. λ the set $\{U_n(\lambda)\}$ is strong-* dense in $\mathcal{U}(\mathcal{C}(\lambda))$. Let $\mathcal{T}$ be that subset of $\mathcal{E}(\mathcal{C})$ consisting of f with finite support contained in the set $\{U_n\}$ whose values $f(U_n)$ are rational. Clearly, $\mathcal{T}$ is countable. Without loss of generality we may assume $A \in \mathcal{C}_1$; thus $A(\lambda) \in \mathcal{C}(\lambda)_1$ and $\overline{CO}(A(\lambda))$ is contained in S μ-a.e. Since $B(\lambda) \in C(A(\lambda))$ μ-a.e., there exists a Borel null set Λ_0 such that $B(\lambda) \in \overline{CO}(A(\lambda))$ for $\lambda \notin \Lambda_0$. Put $\tilde{\Lambda} = \Lambda - \Lambda_0$. By Lusin's lemma, $\tilde{\Lambda}$ differs by a Borel null set from the union of a countable sequence of increasing closed subsets $\{a_r\}$ such that $A(\lambda)$, $B(\lambda)$ and the $U_n(\lambda)$ are strong-* continuous on each set a_r. By [4, p. 228], each set a_r differs by a Borel null set from the union of a countable sequence of increasing compact subsets $\{K_n^r\}$. Since

$$(f \cdot A)(\lambda) = \sum_{k=1}^{p} f(V_k) V_k(\lambda) A(\lambda) V_k(\lambda)^*,$$

where $V_1, \ldots, V_p$ is the support of f, $(f \cdot A)(\lambda)$ is also strong-* continuous for each $f \in \mathcal{T}$ on each set K_n^r.

We now choose some K_n^r fixed, but arbitrary for the next portion of our argument. Since $B(\lambda) \in \overline{CO}(A(\lambda))$ for each $\lambda \in K_n^r$, there are $f \in \mathcal{T}$ such that

$$W((f \cdot A)(\lambda) - B(\lambda)) < 1/m$$

for any integer $m > 0$ and any $\lambda \in K_n^r$. By continuity, each such inequality holds on an open subset of K_n^r and by compactness, there exists a finite covering of K_n^r by such sets. Hence, there are disjoint Borel sets $F_1, \ldots, F_L$ such that $K_n^r = \cup_{i=1}^{L} F_i$ and there are f_i contained in $\mathcal{T}$ such that $W((f_i \cdot A)(\lambda) - B(\lambda)) < 1/m$ for $\lambda \in F_i$. Note that the following construction can be extended to any finite L, but in order to simplify the notation, we shall assume $L = 2$. Accordingly, put $K_n^r = F \cup G$, where $F = F_1$ and $G = F_2$. Let f and g denote f_1 and f_2 respectively. Let $V_1, \ldots, V_p$ be the support of f and let $W_1, \ldots, W_q$ be the support of g. Then from [4, Lemma I.4.11], Lemma 2 and P.M.C., we may define unitaries $U_{ij} = \int_{\Lambda} \oplus U_{ij}(\lambda) \mu(d\lambda) \in \mathcal{C}$ by

$$U_{ij}(\lambda) = \begin{cases} V_i(\lambda), & \text{if } \lambda \in F, \\ W_j(\lambda), & \text{if } \lambda \in G. \end{cases}$$

Define t with support on U_{ij} by $t(U_{ij}) = f(V_i)g(W_j)$. Then

$$\sum_{i=1}^{p} \sum_{j=1}^{q} t(U_{ij}) = \sum_{i=1}^{p} \sum_{j=1}^{q} f(V_i)g(W_j)$$

$$= \sum_{p=1}^{p} f(V_i) \sum_{j=1}^{q} g(W_j) = 1,$$

so that $t \in \mathcal{S}(\mathcal{Q})$. Thus

$$W((t \cdot A)(\lambda) - B(\lambda)) = W\left(\sum_{i=1}^{p} \sum_{j=1}^{q} t(U_{ij}) U_{ij}(\lambda) A(\lambda) U_{ij}(\lambda)^* - B(\lambda) \right)$$

$$= \left\{ \begin{array}{ll} W\left(\sum_{i=1}^{p} f(V_i) V_i(\lambda) A(\lambda) V_i(\lambda)^* - B(\lambda) \right), & \text{if } \lambda \in F, \\[2em] W\left(\sum_{j=1}^{q} g(W_j) W_j(\lambda) A(\lambda) W_j(\lambda)^* - B(\lambda) \right), & \text{if } \lambda \in G \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} W((f \cdot A)(\lambda) - B(\lambda)), & \text{if } \lambda \in F, \\ W((g \cdot A)(\lambda) - B(\lambda)), & \text{if } \lambda \in G \end{array} \right\} < 1/m \quad \text{for } \lambda \in K_n^r.$$

It follows by diagonalization that there exists a countable sequence of functions $\{t_m^{r,n}\}$ contained in $\mathcal{S}(\mathcal{Q})$ such that $\{W((t_m^{r,n} \cdot A)(\lambda) - B(\lambda))\}$ converges to zero for each $\lambda \in K_n^r$.

Since K_n^r was chosen arbitrarily, each of the sets K_n^r may be treated similarly, and we can extract from $\bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \{t_m^{r,n}\}$ a countable sequence $\{t_m = t_m^{m,m}\}$ contained in $\mathcal{S}(\mathcal{Q})$ such that $\{(t_m \cdot A)(\lambda) - B(\lambda)\}$ converges strong-* and hence strongly to zero μ-a.e. By [4, Lemma I.3.7] and [3, Corollary III.6.13], it follows that $\{((t_m \cdot A)(\lambda) - B(\lambda))x\}$ converges to zero in μ-measure for each $x \in h_\infty$. Thus $\{t_m \cdot A\}$ converges strongly to B by [4, Lemma I.3.6].   Q.E.D.

LEMMA 5. *Let $\mathcal{Q}$ be a factor acting in h and suppose $\mathcal{Q}$ is of type* I *or* II. *Then $E \in \mathcal{F}(\mathcal{Q})$ iff for some positive integer n, h contains a finite set of unit vectors $x_1, \ldots, x_n$ such that $Ex_i = x_i$ for $i = 1, \ldots, n$ and such that $\sum_{i=1}^{n} ((EAE)(EBE)x_i, x_i) = \sum_{i=1}^{n} ((EBE)(EAE)x_i, x_i)$ for each pair $A, B \in \mathcal{Q}_1$.*

PROOF. That such a set of vectors exists whenever E is finite can be ascertained by applying [4, Lemma II.3.13] to the finite factor $\mathcal{Q}_E = \{(EAE|Eh)|A \in \mathcal{Q}\}$ when $\mathcal{Q}$ is infinite, and to $\mathcal{Q}$ itself otherwise.

To prove the converse, we suppose E is infinite. Then there exists a projection $F \leqslant E$, $F \neq E$ such that $F \sim E$.[3] By [4, Corollary II.1.10], we may write $E = \sum_{m=1}^{\infty} E_m$, where the projections $E_m \in \mathcal{Q}$ are mutually orthogonal and each E_m is equivalent to F. Put $V = E_1$ and $W = \sum_{m=2}^{\infty} E_m$. Then by [4, Corollary II.1.2], $V \sim E$ and $\sum_{m=2}^{\infty} E_m \sim \sum_{m=1}^{\infty} E_m$. Thus $V \sim E \sim W$ and there exist partial isometries $P, Q \in \mathcal{Q}$ with the following properties:

$$P: Eh \to Vh, \quad V = PP^*, \quad E = P^*P, \quad EPE = P, \quad EP^*E = P^*;$$
$$Q: Eh \to Wh, \quad W = QQ^*, \quad E = Q^*Q, \quad EQE = Q, \quad EQ^*E = Q^*.$$

Thus

$$\sum_{i=1}^{n} (Ex_i, x_i) = \sum_{i=1}^{n} (P^*Px_i, x_i) = \sum_{i=1}^{n} ((EP^*E)(EPE)x_i, x_i)$$

$$= \sum_{i=1}^{n} ((EPE)(EP^*E)x_i, x_i) = \sum_{i=1}^{n} (PP^*x_i, x_i) = \sum_{i=1}^{n} (Vx_i, x_i),$$

---

[3] $E \sim F$ (*E is equivalent to* F) iff there is a partial isometry $P \in \mathcal{Q}$ such that $PP^* = E$ and $P^*P = F$.

so that $\sum_{i=1}^{n} ((E - V)x_i, x_i) = 0$, and

$$0 < n = \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} (x_i, x_i)$$

$$= \sum_{i=1}^{n} (Ex_i, x_i) = \sum_{i=1}^{n} (Q^*Qx_i, x_i) = \sum_{i=1}^{n} ((EQ^*E)(EQE)x_i, x_i)$$

$$= \sum_{i=1}^{n} (QQ^*x_i, x_i) = \sum_{i=1}^{n} (Wx_i, x_i) = \sum_{i=1}^{n} ((E - V)x_i, x_i) = 0.$$

This contradiction proves that E is finite.   Q.E.D.

LEMMA 6. *Let* $\mathcal{C} = \int_\Lambda \oplus \mathcal{C}(\lambda)\mu(d\lambda)$ *be a* W^*-*algebra acting in* h *and suppose* $\mathcal{C}$ *is of type* I *or* II. *Then if* N *is a Borel subset of* Λ, *the set* $F = \{(\lambda, E)|\lambda \in N, E \in \mathcal{F}(\mathcal{C}(\lambda))\}$ *is an analytic subset of* $\Lambda \times S$.

PROOF. Let the countable sequence of operators $\{B_m\}$ be as in Lemma 1 and let $\mathbb{S}$ denote the unit shell of h_∞ taken with its metric topology. Let the positive integer n be given and let $\mathbb{S}^{(n)}$ be the n-fold Cartesian product of $\mathbb{S}$ with itself. Then $\mathbb{S}$, $\mathbb{S}^{(n)}$ and $S \times \mathbb{S}^{(n)}$ are complete separable metric spaces.

Define subsets $F(i, n)$ of $\Lambda \times S \times \mathbb{S}^{(n)}$ as sets of all $(n + 2)$-tuples $(\lambda, E, x_1, \ldots, x_n)$ satisfying the following conditions:

(a) $\lambda \in N \cap e_i$,

(b) $E \in \mathcal{C}(\lambda) \cap S$,

(c) $E^* = E = E^2$,

(d) $Ex_k = x_k, k = 1, \ldots, n$,

(e) $\sum_{k=1}^{n} ((EB_m(\lambda)E)(EB_j(\lambda)E)x_k, x_k) = \sum_{k=1}^{n} ((EB_j(\lambda)E)(EB_m(\lambda)E)x_k, x_k)$ for all m, j.

Condition (a) defines a Borel set. By Lemma 1, conditions (a) and (b) define a Borel set. Conditions (c) define a closed set. Condition (d) defines a closed set and by the strong-* continuity of the operator valued functions $B_m(\lambda)$, the strong-* bicontinuity of multiplication on bounded sets and the continuity of the inner product, each of the conditions (e) defines a closed set as well. Thus $F(i, n)$ is a Borel subset of $\Lambda \times S \times \mathbb{S}^{(n)}$.

Since each $A \in \mathcal{C}(\lambda)_1$ is the strong-* limit of a countable sequence contained in the set $\{B_m(\lambda)\}$, it is clear that the set $F(i, n)$ is identical to the set of all $(n + 2)$-tuples $(\lambda, E, x_1, \ldots, x_n)$ satisfying the above conditions (a) through (d) and the following condition (e'):

(e') $\sum_{k=1}^{n} ((EAE)(EBE)x_k, x_k) = \sum_{k=1}^{n} ((EBE)(EAE)x_k, x_k)$ for all $A, B \in \mathcal{C}(\lambda)_1$.

Finally, conditions (c) show that E is a projection while conditions (d) and (e') show by Lemma 5 that E is nonzero and finite.

Let π denote the natural projection of $\Lambda \times S \times \mathbb{S}^{(n)}$ into $\Lambda \times S$. By [4, Lemma I.4.4], $\pi(F(i, n))$ is an analytic subset of $\Lambda \times S$. Hence, $F = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \pi(F(i, n))$ is analytic by [4, Lemma I.4.2].   Q.E.D.

LEMMA 7. *Let* $\mathcal{Q} = \int_\Lambda \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ *be a W^*-algebra acting in h and let E, $F \in \mathcal{Q}$.*
Then $E = \int_\Lambda \oplus E(\lambda)\mu(d\lambda)$, $F = \int_\Lambda \oplus F(\lambda)\mu(d\lambda)$ *and,*

(1) $E \in \mathcal{P}(\mathcal{Q})$ *iff* $E(\lambda) \in \mathcal{P}(\mathcal{Q}(\lambda))$ *μ-a.e.,*

(2) E *is Hermitian iff* $E(\lambda)$ *is Hermitian μ-a.e.,*

(3) *if E and F are Hermitian, then* $E \leqslant F$ *iff* $E(\lambda) \leqslant F(\lambda)$ *μ-a.e.,*

(4) *if* E, $F \in \mathcal{P}(\mathcal{Q})$, *then* $E \sim F$ *iff* $E(\lambda) \sim F(\lambda)$ *μ-a.e.*

PROOF. The decomposability of all $A \in \mathcal{Q}$ is established by [**4**, Lemma I.5.2] and the validity of (1) and (2) is an immediate consequence of [**4**, Lemmas I.5.2 and I.3.1].

We shall now prove (3). An operator $T \in \mathcal{Q}$ is Hermitian and nonnegative iff $T = AA^*$, where $A \in \mathcal{Q}$ (see [**4**, pp. 247–248]). Thus by (2) and [**4**, Lemma I.3.1], it follows that $T \geqslant 0$ iff $T(\lambda) \geqslant 0$ μ-a.e. Put $T = F - E$. Then by [**4**, Lemma I.3.1], $T(\lambda) = F(\lambda) - E(\lambda)$ μ-a.e. and it follows that $E \leqslant F$ iff $E(\lambda) \leqslant F(\lambda)$ μ-a.e.

To prove (4) we argue as follows. Suppose $E \sim F$. Then there exists a partial isometry $P = \int_\Lambda \oplus P(\lambda)\mu(d\lambda)$ such that $PP^* = E$ and $P^*P = F$. By [**4**, Lemma I.3.1], $P(\lambda)P(\lambda)^* = E(\lambda)$ μ-a.e. and $P(\lambda)^*P(\lambda) = F(\lambda)$ μ-a.e. By (1), $E(\lambda)$, $F(\lambda) \in \mathcal{P}(\mathcal{Q}(\lambda))$ μ-a.e., so that $P(\lambda)$ is a partial isometry μ-a.e. and we have $E(\lambda) \sim F(\lambda)$ μ-a.e.

Conversely, suppose $E(\lambda) \sim F(\lambda)$ μ-a.e. By Lusin's lemma, Λ differs by a Borel null set from the union of a countable sequence of increasing closed subsets $\{a_i\}$ such that $E(\lambda)$ and $F(\lambda)$ are strong-* continuous on each set a_i. Define subsets $G(i)$ of $\Lambda \times S$ as sets of all pairs (λ, P) satisfying the following condtions:

(a) $\lambda \in a_i$,

(b) $P \in \mathcal{Q}(\lambda) \cap S$,

(c) $(P^*P)^2 = P^*P$, $(PP^*)^2 = PP^*$,

(d) $P^*P = E(\lambda)$, $PP^* = F(\lambda)$.

Conditions (a) and (c) define closed sets. By Lemma 1, conditions (a) and (b) define a Borel set. Each of the conditions (d) defines a closed set. Thus $G(i)$ is a Borel subset of $\Lambda \times S$ and so is $G = \cup_{i=1}^\infty G(i)$. By [**4**, Lemma I.4.3], G is analytic. By P.M.C., there exists a Borel subset F_1 of $\pi(G)$ with positive measure and a μ-measurable mapping y of F_1 into S such that $(\lambda, y(\lambda)) \in G$ for each $\lambda \in F_1$. Put $y(\lambda) = 0$ for $\lambda \notin F_1$ and define μ-measurable operator valued functions $P(\lambda)$ by $P(\lambda) = y(\lambda)$. Then by [**4**, Definition I.2.5], we may write $P = \int_\Lambda \oplus P(\lambda)\mu(d\lambda)$ and by [**4**, Lemmas I.3.1 and I.5.2], we know that $PP^* = E$, $P^*P = F$ and $P \in \mathcal{Q}$. Thus P is a partial isometry and $F \sim E$. Q.E.D.

LEMMA 8. *Let* $\mathcal{Q} = \int_\Lambda \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ *be a W^*-algebra acting in h and suppose $\mathcal{Q}$ is of type* I *or* II. *Then there exists a countable sequence* $\{E_n\}$ *contained in* $\mathcal{F}(\mathcal{Q})$ *such that for μ-a.a. λ the set* $\{E_n(\lambda)\}$ *is strong-* dense in* $\mathcal{P}(\mathcal{Q}(\lambda))$.

PROOF. Define subsets $G(i)$ of $\Lambda \times S$ as sets of all pairs (λ, E) satisfying the following conditions:

(a) $\lambda \in e_i$,

(b) $E \in \mathcal{F}(\mathcal{Q}(\lambda))$.

By Lemma 6 each $G(i)$ is an analytic set. Since S is a separable metric space, there exists a countable sequence of open sets $\{W_n\}$ contained in S which forms a basis for the topology on S. Let $F(i, n) = \{(\lambda, E) \in G(i) \mid E \in W_n\}$. Clearly, each $F(i, n)$ is an analytic set and so is $F(n) = \bigcup_{i=1}^{\infty} F(i, n)$. By P.M.C., there exists a countable sequence of Borel null sets $M(n)$ and a countable sequence of μ-measurable mappings y_n of $H(n) = \pi(F(n)) - M(n)$ into S such that $(\lambda, y_n(\lambda)) \in F(n)$ for each $\lambda \in H(n)$. Put $y_n(\lambda) = 0$ for $\lambda \notin H(n)$ and define μ-measurable operator valued functions $E_n(\lambda)$ by $E_n(\lambda) = y_n(\lambda)$. Then by [4, Definition I.2.5], we may write $E_n = \int_\Lambda \oplus E_n(\lambda)\mu(d\lambda)$.

We assert that $\{E_n\}$ is contained in $\mathcal{F}(\mathcal{Q})$. We know that $\{E_n\}$ is contained in $\mathcal{P}(\mathcal{Q})$ by Lemma 7 and it remains to show that E_n is finite for each n. To prove this, we assume E_n is infinite for some n. Then there exists a projection $E \leqslant E_n$, $E \neq E_n$ such that $E \sim E_n$. Also by Lemma 7, $E(\lambda) \leqslant E_n(\lambda)$ and $E(\lambda) \sim E_n(\lambda)$ μ-a.e. By [4, Definition I.2.5], $E \neq E_n$ implies that there exists a set N of positive measure such that $E(\lambda) \neq E_n(\lambda)$ for $\lambda \in N$. Thus $E_n(\lambda)$ is infinite on N. This contradiction proves E_n is finite for each n.

Now we shall show that $\{E_n(\lambda)\}$ is strong-* dense in $\mathcal{P}(\mathcal{Q}(\lambda))$ μ-a.e. Since $\mathcal{P}(\mathcal{Q}(\lambda))$ is contained in S, the set $\{K_n(\lambda) = \mathcal{P}(\mathcal{Q}(\lambda)) \cap W_n\}$ is a basis for $\mathcal{P}(\mathcal{Q}(\lambda))$. Let $D = e \cup \bigcup_{n=1}^{\infty} M(n)$ and suppose $\lambda \notin D$. Then if $K_n(\lambda)$ is not empty, $\lambda \in H(n) - D$ and $E_n(\lambda) \in K_n(\lambda)$ also. It follows that for each $\lambda \in H(n) - D$, $E_n(\lambda) \in K_n(\lambda)$ so that the sequence $\{E_n(\lambda)\}$ is strong-* dense in $\mathcal{P}(\mathcal{Q}(\lambda))$ μ-a.e.   Q.E.D.

3. Statements and proofs of main theorems.

THEOREM 9. *Let $\mathcal{Q}$ be a W^*-algebra of type* I *or* II *acting in h. Then $\mathcal{K}(\mathcal{Q})$ is strongly dense in $\mathcal{Q}$.*

PROOF. Let $\mathcal{Q} = \int_\Lambda \oplus \mathcal{Q}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of $\mathcal{Q}$ into factors and let $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda) \in \mathcal{Q}$. We shall show that there exists a countable sequence of operators $\{A_m = \int_\Lambda \oplus A_m(\lambda)\mu(d\lambda)\}$ contained in $\mathcal{K}(\mathcal{Q})$ which converges strongly to A. By Lemma 8, there is a countable sequence $\{E_n = \int_\Lambda \oplus E_n(\lambda)\mu(d\lambda)\}$ contained in $\mathcal{F}(\mathcal{Q})$ such that the set $\{E_n(\lambda)\}$ is strong-* dense in $\mathcal{P}(\mathcal{Q}(\lambda))$ μ-a.e. By the compactness of Λ we can apply diagonalization, as in Lemma 4, to demonstrate that $\{E_n\}$ has a subsequence denoted by $\{B_m = \int_\Lambda \oplus B_n(\lambda)\mu(d\lambda)\}$ for which $\{B_m(\lambda)\}$ converges strongly to the identity in $\mathcal{Q}(\lambda)$ μ-a.e. Define the sequence $\{A_m\}$ by $A_m = AB_m$. Then for each m, A_m is decomposable and $A_m(\lambda) = A(\lambda)B_m(\lambda)$ μ-a.e. by [4, Lemma I.3.1]. Since $\mathcal{F}(\mathcal{Q}(\lambda))$ is contained in $\mathcal{C}_1(\mathcal{Q}(\lambda))$ and since $\mathcal{C}_1(\mathcal{Q}(\lambda))$ is an ideal in $\mathcal{Q}(\lambda)$ (see [4, Theorem II.3.12]), we have $A_m(\lambda) \in \mathcal{C}_1(\mathcal{Q}(\lambda))$ μ-a.e. for each m, so that $\{A_m\}$ is contained in $\mathcal{K}(\mathcal{Q})$.

Now we shall show that $\{A_m\}$ is the required sequence. Since $|A_m| = |AB_m| \leqslant |A||B_m| = |A|$ for each m, the sequence $\{A_m\}$ is uniformly bounded; thus $\sup_m |A_m| \leqslant |A| < \infty$. For each $x \in h_\infty$,

$$|(A_m(\lambda) - A(\lambda))x| = |(A(\lambda)B_m(\lambda) - A(\lambda))x| = |A(\lambda)(B_m(\lambda) - I(\lambda))x|$$

$$\leqslant |A(\lambda)||(B_m(\lambda) - I(\lambda))x| \leqslant |A||(B_m(\lambda) - I(\lambda))x| \mu\text{-a.e.}$$

and, since $\{B_m(\lambda)\}$ converges strongly to the identity in $\mathcal{C}(\lambda)$, μ-a.e., it follows that $\{A_m(\lambda)\}$ converges strongly to $A(\lambda)$ μ-a.e. Also $\{A_m(\lambda)x\}$ converges to $A(\lambda)x$ in μ-measure for each $x \in h_\infty$ by [4, Lemma I.3.7] and [3, Corollary III.6.13]. Thus $\{A_m\}$ converges strongly to A by [4, Lemma I.3.6].   Q.E.D.

THEOREM 10. *Let $\mathcal{C}$ be a W^*-algebra of type I_∞ or II_∞ acting in h and let $A \in \mathcal{C}$. Then $C(A) = \{0\}$ iff $A \in \mathcal{K}(\mathcal{C})$.*

PROOF. Let $\mathcal{C} = \int_\Lambda \oplus \mathcal{C}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of $\mathcal{C}$ into factors of type I_∞ or II_∞ and let $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$. Suppose $C(A) = \{0\}$. We shall untimately show that $A \in \mathcal{K}(\mathcal{C})$, but we must first prove $C(A(\lambda)) = \{0\}$ μ-a.e. Let $\{U_n\}$ and $\mathcal{T}$ be as in the proof of Lemma 4. Let $\{b_n\}$ be a countable sequence of scalars that is dense in the unit ball of the complex numbers and let d denote the metric which defines the topology on S. We may assume without loss of generality that $A \in \mathcal{C}_1$; thus $A(\lambda) \in \mathcal{C}(\lambda)_1$ and $C(A(\lambda))$ is contained in S μ-a.e. By Lusin's lemma, Λ differs by a Borel null set from the union of a countable sequence of increasing closed subsets $\{a_i\}$ such that $A(\lambda)$ and the $U_n(\lambda)$ are strong-* continuous on each set a_i. To prove our assertion, we shall show that the set $N = \{\lambda | \lambda \in \bigcup_{i=1}^\infty a_i, C(A(\lambda)) \neq \{0\}\}$ contains a μ-measurable set of positive measure and that this implies the existence of a nonzero element in $C(A)$.

Define subsets $E(i, f, m, n)$ of $\Lambda \times S$ as sets of all pairs (λ, T) satisfying the following conditions:

(a) $\lambda \in a_i$

(b) $T = b_n I$,

(c) $d((f \cdot A)(\lambda), T) \leqslant 1/m$.

Each of the conditions (a) and (b) defines a closed set. Since $(f \cdot A)(\lambda) = \sum_{k=1}^p f(V_k)V_k(\lambda)A(\lambda)V_k(\lambda)^*$, where $V_1, \ldots, V_p$ is the support of f, $(f \cdot A)(\lambda)$ is also strong-* continuous on each set a_i so that condition (c) defines a closed set also. Thus $E(i, f, m, n)$ is a Borel subset of $\Lambda \times S$ and so is

$$E = \bigcup_{i=1}^\infty \ \bigcup_{m=1}^\infty \ \bigcup_{n=1}^\infty \ \bigcup_{f \in \mathcal{T}} E(i, f, m, n).$$

By [4, Lemma I.4.3], E is analytic.

By P.M.C., there exists a Borel subset F_1 of $\pi(E)$ with positive measure and a μ-measurable mapping y of F_1 into S such that $(\lambda, y(\lambda)) \in E$ for each $\lambda \in F_1$. Put $y(\lambda) = 0$ for $\lambda \notin F_1$ and define μ-measurable operator valued functions $B(\lambda)$ by $B(\lambda) = y(\lambda)$. Then by [4, Definition I.2.5], we may write $B = \int_\Lambda \oplus B(\lambda)\mu(d\lambda)$. Clearly, $B(\lambda) \neq 0$ for $\lambda \in F_1$. Thus $B \neq 0$ by construction and since $B(\lambda) \in C(A(\lambda))$ μ-a.e., we have $B \in C(A)$ by Lemma 4. This contradiction proves that $C(A(\lambda)) = \{0\}$ μ-a.e. Hence by [1, Theorem 3], $A(\lambda) \in \mathcal{C}_1(\mathcal{C}(\lambda))$ μ-a.e. Thus $A \in \mathcal{K}(\mathcal{C})$ by definition.

Conversely, suppose $A \in \mathcal{K}(\mathcal{C})$; thus $A(\lambda) \in \mathcal{C}_1(A(\lambda))$ μ-a.e. Then by [1, Lemma 6], $C(A(\lambda)) = \{0\}$ μ-a.e. and it follows that $C(A) = \{0\}$ by Lemma 4.   Q.E.D.

THEOREM 11. *Let $\mathcal{C}$ be a W^*-algebra of type III acting in h and let $A \in \mathcal{C}$. Then $C(A) = \{0\}$ iff $A \in \mathcal{K}(\mathcal{C})$.*

PROOF. We note that $C(A) = \{0\}$ whenever $A = 0$ is obvious by the definition of $C(A)$.

Conversely, let $C(A) = \{0\}$ and suppose $A \neq 0$. Let $\mathcal{C} = \int_\Lambda \oplus \mathcal{C}(\lambda)\mu(d\lambda)$ be the direct integral decomposition of $\mathcal{C}$ into factors of type III and let $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$. Then there exist Borel null sets L and M such that $\mathcal{C}(\lambda)$ is a type III factor for $\lambda \notin L$ and such that $A(\lambda) \in \mathcal{C}(\lambda)$ for $\lambda \notin M$. By [4, Definition I.2.5], $A \neq 0$ implies the existence of a Borel set N of positive measure such that $A(\lambda) \neq 0$ for $\lambda \in N$. We may assume without loss of generality that $A \in \mathcal{C}_1$; thus $A(\lambda) \in \mathcal{C}(\lambda)_1$ μ-a.e. By [1, Theorem 3], $C(A(\lambda))$ is the closure of the set $\{(A(\lambda)x, x) \mid x \in h_\infty, |x| = 1\}$ for $\lambda \notin L \cup M$. By Lusin's lemma, Λ differs by a Borel null set from the union of a countable sequence of increasing closed subsets $\{a_i\}$ such that $A(\lambda)$ is strong-* continuous on each set a_i. Let $\mathcal{S}$ be as in the proof of Lemma 6.

Define subsets $E(i, m)$ of $\Lambda \times \mathcal{S}$ as sets of all pairs (λ, x) satisfying the following conditions:

(a) $\lambda \in (N - L \cup M) \cap a_i$,

(b) $|(A(\lambda)x, x)| > 1/m$.

Condition (a) defines a Borel set. Condition (b) defines an open set. Thus $E(i, m)$ is a Borel subset of $\Lambda \times \mathcal{S}$ and so is $E = \bigcup_{i=1}^\infty \bigcup_{m=1}^\infty E(i, m)$. By [4, Lemma I.4.3], E is analytic. By P.M.C., there exists a Borel subset F_1 of $\pi(E)$ with positive measure and a μ-measurable mapping y of F_1 into $\mathcal{S}$ such that $(\lambda, y(\lambda)) \in E$ for each $\lambda \in F_1$. Put $y(\lambda) = \hat{x}$ for $\lambda \notin F_1$, where $\hat{x}$ is a fixed element of h_∞ with $|\hat{x}| = 1$.

By [4, Lemma I.3.7], $A(\lambda)x$ is a μ-measurable function of λ for each $x \in h_\infty$. Then by the continuity of the inner product and the Schwarz inequality, it follows that $(A(\lambda)x, x)$ is a bounded μ-measurable scalar valued function of λ for each $x \in h_\infty$. Define μ-measurable operator valued functions $B(\lambda)$ by $B(\lambda) = (A(\lambda)y(\lambda), y(\lambda))I$. Then by [4, Definition I.2.5], we may write $B = \int_\Lambda \oplus B(\lambda)\mu(d\lambda)$. Clearly, $B(\lambda) \neq 0$ for $\lambda \in F_1$. Thus $B \neq 0$ by construction and since $B(\lambda) \in C(A(\lambda))$ for μ-a.a. λ, we have $B \in C(A)$ by Lemma 4. This contradiction proves $A = 0$. Since $\mathcal{K}(\mathcal{C}) = \{0\}$ for W^*-algebras of type III, our proof is complete.   Q.E.D.

The following definition is from [2, p. 284].

DEFINITION 12. Let $\mathcal{C}$ be a W^*-algebra and let G be a subgroup of $\mathcal{U}(\mathcal{C})$. By a *Schwartz map* relative to $(G, \mathcal{C})$ one means a linear map P of $\mathcal{C}$ into itself such that:

(a) $P(A) = UP(A)U^*$ for all $U \in G$ and all $A \in \mathcal{C}$,

(b) $P(A) \in \overline{CO}_G(A)$, where $\overline{CO}_G(A)$ is the weak closure of the convex hull generated by elements of the type UAU^* as U ranges over G.

COROLLARY 13. *Let $\mathcal{C}$ be a W^*-algebra of type* I_∞, II_∞, *or* III *acting in h and let $\mathcal{B}$ and $\mathcal{Z}(\mathcal{B})$ be W^*-subalgebras of $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ respectively. If G is a subgroup of $\mathcal{U}(\mathcal{C})$ containing $\mathcal{U}(\mathcal{B})$ and P is a Schwartz map relative to $(G, \mathcal{C})$ of $\mathcal{C}$ into $\mathcal{B}$, then P annihilates $\mathcal{K}(\mathcal{C})$.*

PROOF. We shall first show that $\overline{CO}_{\mathcal{U}(\mathcal{B})}(P(A))$ is contained in $\overline{CO}_{\mathcal{U}(\mathcal{C})}(A)$ for $A \in \mathcal{C}$ and $P(A) \in \mathcal{B}$. Let $B \in \overline{CO}_{\mathcal{U}(\mathcal{B})}(P(A))$. Then there exists a countable sequence of functions $\{f_n\}$ contained in $\mathcal{S}(\mathcal{B})$ such that $\{f_n \cdot P(A)\}$ converges

strongly and hence weakly to B. Since $\mathfrak{U}(\mathfrak{B})$ is contained in G, we have

$$f_n \cdot P(A) = \sum_{U \in \mathfrak{U}(\mathfrak{B})} f_n(U)UP(A)U^*$$

$$= \sum_{U \in \mathfrak{U}(\mathfrak{B})} f_n(U)P(A) = P(A)$$

for each n. Thus $B = P(A) \in \overline{CO}_G(A)$ which is contained in $\overline{CO}_{\mathfrak{U}(\mathfrak{A})}(A)$. Hence $\overline{CO}_{\mathfrak{U}(\mathfrak{B})}(P(A)) \cap \mathfrak{Z}(\mathfrak{B})$ is contained in $\overline{CO}_{\mathfrak{U}(\mathfrak{A})}(A) \cap \mathfrak{Z}(\mathfrak{A})$. It also follows readily from [**4**, Theorem I.1.3, Corollary I.1.11] and Definition 12 that $P(A) \in \overline{CO}_{\mathfrak{U}(\mathfrak{B})}(P(A)) \cap \mathfrak{Z}(\mathfrak{B})$ for all $A \in \mathfrak{A}$. Now if $A \in \mathfrak{K}(\mathfrak{A})$, then $\overline{CO}_{\mathfrak{U}(\mathfrak{A})}(A) \cap \mathfrak{Z}(\mathfrak{A}) = \{0\}$ by Theorems 10 and 11. Thus $P(A) = 0$ for all $A \in \mathfrak{K}(\mathfrak{A})$.   Q.E.D.

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